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SOME DIFFERENTIAL GEOMETRY MOTIVATED BY COILING TENDRILS

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Abstract. In *On The Movements and Habits of Climbing Plants* [1865], Charles Darwin discussed the phenomenology of tendrils—filamentous growths on certain plants that grasp and bind the plants to supportive structures. Observations more than a century later led to two striking theorems apparently first noticed in [Raugh, *Some Geometry Problems Suggested by the Shapes of Tendrils*, Stanford thesis, 1978]. They are stated and proved here. One of them generalizes the Centroid Volume Theorem of Pappus, and it is extended here for the first time to higher dimensions. It is also shown that the Centroid Surface Theorem of Pappus cannot be generalized similarly. The Conclusion relates this "tendril theory" to the "tubes" of Weyl and Hotelling used to study manifolds.

1. Introduction

The following photograph nicely illustrates tendrils in the cucumber family. It is evident that the regularity of coiling must be due to physical forces of growth and tension.

Figure 1. The spiraling horizontal tendril began as a slightly arched filament; once it attached to a support at the left, it began a process of growth resulting in spring-like coiling. ⟨http://www.biologie.uni-hamburg.de/b-online/e32/32e.htm⟩ (accessed July 28, 2008). Copyright ©2003 Peter v. Sengbusch, used with permission

[1]Thanks to Basil Gordon and Robert Osserman for helpful comments.
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Barely visible in Fig. 1, just to the right of the large vertical stem, is a place where the sense of twisting makes a reversal, a necessity noted by Charles Darwin because both ends of the tendril are fixed in place during growth of the coils. It is a challenging problem of continuum mechanics to account for how such regular spirals are formed in nature. Darwin initiated phenomenological studies more than a hundred years ago [8], and still today scientists continue studying tendrils with modern laboratory techniques and mathematics (e.g., [10]). That problem will not be tackled here. But some geometrical theory motivated by considerations of tendrils will be presented.

It turns out that the theory developed here has some elements in common with an earlier theory of "tubes" initiated by Weyl and Hotelling. The Conclusion spells out this connection with several examples and references.

2. A Mathematical Tendril Model

The tendril model postulated by Raugh [19] stated that a tendril at any given moment of time is characterized by a curve $\mathbf{x}(s)$, referred to below as the *centerline*, running longitudinally from the base of the tendril to its tip in such a way that the centerline passes through every normal section of the tendril at the centroid of the normal section. The model assumes that the normal section of the tendril remains rigidly fixed in shape as the tendril is deformed in both curvature and torsion. In this paper, we require $\mathbf{x}(s)$ to have non-vanishing curvature and to be of class C^3.

Structural components of a tendril can be incorporated in this model. For example, vascular bundles that run longitudinally through a tendril can be modeled as curves called *laterals* that thread through the model in a way that will be specified below. As will be shown, the volume of a tendril modeled in this way is invariant under deformation, and symmetrically opposed laterals, called *twin laterals*, have equal lengths. A more precise definition of the modeled tendril is given in mathematical terms in Sections 4 and 5 below.

Laterals are defined in terms of the Frenet frame of the centerline.[1] A typical lateral line can be specified by an angle $\theta(s)$ and a radial function $r(s)$ that may vary with the arc length s of the centerline measured from its base point. Thus, if the central line is represented as the vector-valued function $\mathbf{x}(s)$ expressed in euclidean coordinates,

[1]The Frenet-Serret formulas are recalled for reference in the following section.

then, in terms of Frenet frame vectors, a lateral line can also be represented in euclidean coordinates by,

$$\mathbf{l}(s) = \mathbf{x}(s) + (r\cos\theta)\mathbf{n}(s) + (r\sin\theta)\mathbf{b}(s) \tag{1}$$

where $\mathbf{n}(s)$ is the principal normal and $\mathbf{b}(s)$ is the binormal of the centerline at s. Both $r = r(s)$ and $\theta = \theta(s)$ can be allowed to vary.

Two special lateral lines are useful for reference: the *ventral line* and *dorsal line*, for which $\theta = 0$ and $\theta = \pi$, respectively, and $r(s)$ in either case is allowed to vary so that the dorsal and ventral lines run along the dorsal and ventral surface of the tendril, as the case may be. For example, if the tendril is circular in cross-section, then $r(s)$ will be constant (and equal) for both the dorsal and ventral lines. A pair of laterals defined by, say, $(r_1(s), \theta_1(s))$ and $(r_2(s), \theta_2(s))$, are said to be twins if, in every normal plane, they are reflections of each other across the principal normal, i.e., if $r_1(s) = r_2(s)$ and $\theta_1(s) = -\theta_2(s)$. The cause of coiling according to Darwin is that the dorsal portion of a tendril grows more rapidly than the ventral portion.

The model is a suggestive way to think about tendrils because it exhibits both *volume invariance* as the curvature and torsion of the centerline are allowed to vary, and *bilateral symmetry* because any pair in a class of twin laterals to be specified will have equal lengths. What makes these features attractive is the plausible idea that if a tendril may be composed primarily of water, then it must be volume invariant or nearly so when disturbed on a short time scale. And with respect to bilateral symmetry, typical cross sections of tendrils do show symmetric placement of vascular structures, and it is an appealing initial hypothesis to think that they occur in twin pairs that grow at the same rate.

Relevant facts about the Frenet Frame are presented in the next section. In Sections 4 and 5 we deal with twin laterals and the volume invariance property of the thesis model, respectively.

3. Frenet Frame

Facts about the Frenet Frame used in subsequent sections appear here. The "dot" notation is used to indicate differentiation with respect to arc length s. When arc length is the independent variable for a rectifiable curve $\mathbf{x} = \mathbf{x}(s)$, then differentiation yields the unit *tangent vector*:

$$\mathbf{t}(s) = \dot{\mathbf{x}} = \frac{d\mathbf{x}(s)}{ds} \tag{2}$$

The tangent vector, based at $\mathbf{x}(s)$, points in the direction of increasing s.

The *principal normal*, also a unit vector based at $\mathbf{x}(s)$, is defined by

$$\dot{\mathbf{t}} = \kappa\mathbf{n}, \quad \kappa = |\dot{\mathbf{t}}|$$

The scalar function $\kappa = \kappa(s)$ is the *curvature* of $\mathbf{x}$ at s. The curvature may be thought of as the rate the tangent vector changes direction as it travels along $\mathbf{x}$ at unit speed. We assume that the curvature does not vanish, and therefore a well-defined $\mathbf{n}$ exists throughout the length of $\mathbf{x}$—a reasonable assumption for tendrils.

Finally, the unit-length *binormal* vector $\mathbf{b}(s)$ completes the three-dimensional orthonormal Frenet Frame based at $\mathbf{x}(s)$, defined by:

$$\mathbf{b}(s) = \mathbf{t} \times \mathbf{n} \tag{3}$$

The derivatives of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are themselves vectors that can be expressed in terms of the Frenet-Serret formulas:[2]

$$\begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{4}$$

The scalar function $\tau = \tau(s)$ defines the *torsion* of $\mathbf{x}$ at s. See Struik [21]. Some authors, Do Carmo [9], define torsion as $-\tau$, but this difference is not important for us.

4. Bilateral Symmetry of the Tendril Model

We have said that the modeled tendril exhibits bilateral symmetry. This means that, viewing the $\mathbf{n}$–$\mathbf{b}$ plane in a right-hand perspective, a *lateral* of radius r and angle θ measured counterclockwise from the $\mathbf{n}$–axis projects onto the local normal plane at $r\cos\theta\,\mathbf{n} + r\sin\theta\,\mathbf{b}$.

So the equation for the lateral can be expressed as

$$\mathbf{L}(s) = \mathbf{x}(s) + r\cos\theta\,\mathbf{n}(s) + r\sin\theta\,\mathbf{b}(s) \tag{5}$$

Let us first constrain θ to a constant value but allow r to vary smoothly as a function of arc length s on $\mathbf{x}$. Then we may compute the rate-of-change in length of the lateral with respect to s by differentiating Eq. (5) as in:

$$\begin{aligned} \dot{\mathbf{L}}(s) &= \dot{\mathbf{x}} + r\cos\theta\,\dot{\mathbf{n}} + \dot{r}\cos\theta\,\mathbf{n} + r\sin\theta\,\dot{\mathbf{b}} + \dot{r}\sin\theta\,\mathbf{b} \\ &= \mathbf{t} + r\cos\theta\,(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot{r}\cos\theta\,\mathbf{n} - r\sin\theta\,(\tau\mathbf{n}) + \dot{r}\sin\theta\,\mathbf{b} \\ &= (1 - \kappa r\cos\theta)\,\mathbf{t} + (-\tau r\sin\theta + \dot{r}\cos\theta)\,\mathbf{n} + (\tau r\cos\theta + \dot{r}\sin\theta)\,\mathbf{b} \end{aligned}$$

[2]In this formulation and elsewhere in this article, vectors are treated as three dimensional horizontal arrays of coordinates, with coordinates in $\mathbb{R}^3$.

Therefore, having assumed θ constant,

$$\begin{aligned}\left|\dot{\mathbf{L}}\right|^2 &= (1-\kappa r\cos\theta)^2 + (-\tau r\sin\theta + \dot{r}\cos\theta)^2 + (\tau r\cos\theta + \dot{r}\sin\theta)^2 \\ &= (1-\kappa r\cos\theta)^2 + (\tau r)^2 + \dot{r}^2\end{aligned}$$

Now releasing the constraint and allowing $\theta(s)$ to vary with s, we can see by inspection of (5) and the final lines of the previous two sets of equations that we must add

$$-2r\sin\theta\,\dot{\theta}\,(-\tau r\sin\theta + \dot{r}\cos\theta) + 2r\cos\theta\,\dot{\theta}\,(\tau r\cos\theta + \dot{r}\sin\theta) + r^2\dot{\theta}^2$$

$$= 2\tau r^2\dot{\theta} + r^2\dot{\theta}^2$$

to obtain the formulation,

$$\left|\dot{\mathbf{L}}\right|^2 = (1-\kappa r\cos\theta)^2 + (\tau r)^2 + \dot{r}^2 + 2\tau r^2\dot{\theta} + r^2\dot{\theta}^2 \tag{6}$$

valid for variable r and θ.

We infer an immediate consequence of Eq. (6) following the definition: A pair of laterals defined by, say, $(\theta_1, r_1(s))$ and $(\theta_2, r_2(s))$, are said to be *twin laterals* if, in every normal plane, they are reflections of each other across the principal normal, i.e., if $\theta_1(s) = -\theta_2(s)$ and $r_1(s) = r_2(s)$. We have:

Theorem 1: *Twin Laterals.* For the case of constant $\theta_1 = -\theta_2$, twin laterals have equal length. For the case of varying $\theta_1(s) = -\theta_2(s)$, twin laterals do not have equal length.

Proof. The theorem is a consequence in Eq. (6) of the evenness of the cosine function, the equality $r_1(s) = r_2(s)$, and, in the case of varying θ, the inequality of $\dot{\theta}_1$ and $\dot{\theta}_2$ (i.e., $\dot{\theta}_1(s) = -\dot{\theta}_2(s)$).

Comment. We emphasize that the Twin Laterals theorem assures us that vascular bundles poised on symmetrically opposite sides of the principal normal of $\mathbf{x}$ have equal length in just the case of constant angles $\theta_1 = -\theta_2$. Thus we can speak of *identical twins* and *un-identical twins*. It is in this sense that we say a modeled tendril is bilaterally symmetric.

5. Volume Preservation in the Tendril Model

We want to show that for a tendril modeled as above, the volume of the tendril between the normal sections at longitudes s_1 and s_2 on the midline of $\mathbf{x}(s)$ is independent of the curvature and torsion of x. It is convenient to take Eq. (5) as a point of departure and regard a point

X in $\mathbb{R}^3$ as parameterized by arc length s and coordinates (ν, β) in the rectangular two-dimensional **b**–**n** frame,

$$X(s, \nu, \beta) = \mathbf{x}(s) + \nu\, \mathbf{n}(s) + \beta\, \mathbf{b}(s) \tag{7}$$

Now we can define a tendril in terms of a mapping $\mathcal{M}$ from a parameter domain $(s, \nu, \beta) \in \mathbf{P}$ to its three-dimensional model $\mathbf{T}$ in $\mathbb{R}^3$:

$$\mathcal{M} \to \mathbf{T} = \mathcal{M}(\mathbf{P}) : \ (s, \nu, \beta) \mapsto X = \mathbf{x}(s) + \nu \mathbf{n}(s) + \beta \mathbf{b}(s) \tag{8}$$

Here it is understood as before that $\mathbf{x}$ is the centerline of the tendril. By definition of the model, the parameters (ν, β) remain fixed under flexure of the tendril in $\mathbb{R}^3$, subject to the constraint that the tendril is not self-overlapping.

We are interested in the volume of the tendril in the intersection with normal planes for $s \in [s_1, s_2]$. So let $\mathbf{P}[s_1, s_2]$ and $\mathbf{T}[s_1, s_2]$ be the corresponding domain restrictions for the variables defining the tendril. Therefore, the specified volume is given by

$$V[s_1, s_2] = \int_{T[s_1,s_2]} dxdydz = \int_{P[s_1,s_2]} J_{\mathcal{M}} d\nu d\beta ds$$

where $J_{\mathcal{M}}$ is the Jacobian of $\mathcal{M}$:

$$\begin{aligned} J_{\mathcal{M}} &= \left| \det \begin{bmatrix} \frac{\partial X}{\partial s} & \frac{\partial X}{\partial \nu} & \frac{\partial X}{\partial \beta} \end{bmatrix}^t \right| \\ &= \det \left| \begin{pmatrix} 1 - \nu\kappa & -\beta\tau & \nu\tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \right| = 1 - \nu\kappa \end{aligned} \tag{9}$$

Note that the condition of admissibility, i.e., non-overlap of normal sections of a tendril, is a global condition stronger than the local condition that the Jacobian $J_{\mathcal{M}} = 1 - \nu\kappa$ not vanish.

To obtain the preceding line we have used the Frenet frame formulas in Eq. (4) to compute $\partial X / \partial s$, and used the product rule $\det(AB) = (\det A)(\det B)$ and the fact that $\det \begin{pmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{pmatrix}^t = 1$ because the Frenet frame vectors are orthonormal. It follows that the volume of the tendril in any admissible deformed position is given by the integral

$$\begin{aligned} V[s_1, s_2] &= \int_{P[s_1,s_2]} J_{\mathcal{M}}\, d\nu d\beta ds = \int_{P[s_1,s_2]} (1 - \nu\kappa)\, d\nu d\beta ds \\ &= \int_{s_1}^{s_2} \left(\int_{\mathbf{P}[s]} d\nu d\beta \right) ds - \int_{s_1}^{s_2} \kappa \left(\int_{\mathbf{P}[s]} \nu d\nu d\beta \right) ds \end{aligned} \tag{10}$$

where $\mathbf{P}[s]$ denotes the normal section of the tendril at s. The last (inner) integral vanishes because the coordinates (ν, β) are measured from the centroid $\mathbf{x}(s)$ of $\mathbf{P}[s]$.[3] Therefore, finally,

$$V[s_1, s_2] = \int_{s_1}^{s_2} \left(\int_{\mathbf{P}[s]} d\nu d\beta \right) ds = \int_{s_1}^{s_2} A(s) ds \tag{11}$$

where $A(s)$ denotes the area of $\mathbf{P}[s]$, the cross-section of the tendril at s. This result is independent of torsion and curvature of the centerline, which proves the assertion, expressed next as a theorem.

Theorem 2: *Volume Invariance.* Admissible deformations of a modeled tendril preserve volume. The volume is given by Eq. (11).

Comment. Any smooth deformation of a tendril is admissible if it does not cause an overlap between any two normal cross sections of the tendril. Geometers will recognize that the Centroid Volume Theorem of Pappus, also known as the Second Theorem of Pappus, is a special case of Eq. (11).[4]

The Volume Invariance theorem is reminiscent of the *tube formulas* of Weyl and Hotelling, which in one three-dimensional case is equivalent to Eq. (11) for a constant annular normal section. Gray [11] devotes a treatise to applications of tubes in manifold theory. More is said about this in the Conclusion.[5]

Taking into account the comment of Footnote 5, which in effect says that the inner integral of Eq. (10) will vanish for *any* plane area whose centroid lies on the binormal of $\mathbf{x}$ at s, we see that the proof given for Theorem 2 actually proves the last of our theorems:

Theorem 3: *Extended Cavalieri Principle.*

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[3] The inner integral shows that we do not need the full strength of the hypothesis that the centerline pass through the centroid of the cross-section; it is sufficient to require that the first moment about the $\mathbf{b}$–axis vanish for all $s \in (s_1, s_2)$.

[4] Centroid Theorems of Pappus: "The First Theorem states that the surface area A of a surface of revolution generated by rotating a plane curve C about an axis external to C and on the same plane is equal to the product of the arc length s of C and the distance d traveled by its geometric centroid....The Second Theorem states that the volume V of a solid of revolution generated by rotating a plane figure F about an external axis is equal to the product of the area A of F and the distance d traveled by its geometric centroid." [Wikipedia, ⟨`http://en.wikipedia.org/wiki/Pappus's_centroid_theorem`⟩]

[5] I would like to thank Robert Osserman for mentioning the connection between the Volume Invariance Theorem and Weyl's and Hotelling's tube formulas—once in 1978 and once again in 2008—and Nathaniel Grossman for referring me to Gray [11] in a conversation about tubes in 2008.

If at each cross-section of the rectifiable curve $\mathbf{x}(s),\ s \in (s_1, s_2)$ there is a measurable set of points $\mathbf{A}(s)$ with area $A(s)$ and with centroid aligned on the binormal $\mathbf{b}(s)$, and further, if the union of all sets $\mathbf{A}(s)\ s \in (s_1, s_2)$ is measurable, then the volume of the resulting "tube" is

$$\int_{s_1}^{s_2} A(s)ds$$

Comment. Not only does Theorem 3 generalize Theorem 2, we may say that it generalizes *Cavalieri's Principle* as well as the Second Theorem of Pappus.[6]

It is natural at this point to ask whether the First Theorem of Pappus can be generalized in a similar way. That this is impossible is demonstrated in Appendix 2.

Note that for Theorems 1–3, we have assumed that $\mathbf{x}$ is a C^3 curve and that a well-defined Frenet frame exists at all points of $\mathbf{x}$. We have also assumed that normal sections of a tendril do not overlap.

6. Volume Preservation in Higher Dimensions

The proof of Theorems 2 and 3 extends immediately to higher dimensions. Suppose $\mathbf{x}(s)$ is a unit-speed curve of class $\mathbf{C}^3$ in $\mathbb{R}^n$ with non-vanishing curvature, and $\mathbf{T}(s)$ is a surrounding tendril, i.e., an integrable volume with the centroid of its normal section at $\mathbf{x}(s)$ for every s.[7]

Analogous to curves in $\mathbb{R}^3$, we define the *tangent* to $\mathbf{x}$ at s (arc length) as

$$\mathbf{t}(s) = \frac{d}{ds}\mathbf{x}(s)$$

and the *principal normal* at $\mathbf{x}(s)$ by

$$\dot{\mathbf{t}} = \kappa \mathbf{n} \quad k = ||\dot{\mathbf{t}}|| \neq 0$$

Also, in analogy with the case in $\mathbb{R}^3$, we define the *osculating plane* at $\mathbf{x}(s)$ as the span of $\{\mathbf{t}(s), \mathbf{n}(s)\}$ in $\mathbb{R}^n$. And we define the *binormal co-space* at $\mathbf{x}(s)$ as the orthogonal complement of the osculating plane at $\mathbf{x}(s)$.

[6] Cavalieri's Principle: "Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes." [Wikipedia, ⟨http://en.wikipedia.org/wiki/Cavalieri's_principle⟩]

[7] The normal plane to a rectifiable curve $\mathbf{x}$ at s is the hyperplane orthogonal to the unit tangent vector $\mathbf{t}(s)$ of $\mathbf{x}$ at s.

For a given $s = s_0$, there exists an orthonormal basis $\mathfrak{B}(s_0)$, say

$$\mathfrak{B}(s_0) = \{\mathbf{b}_3(s_0), \ldots, \mathbf{b}_n(s_0)\}$$

for the binormal co-space. Therefore, the set

$$\mathfrak{F}(s_0) = \{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}_3(s_0), \ldots, \mathbf{b}_n(s_0)\}$$

constitutes an orthonormal basis for $\mathbb{R}^n$.

Lemma (1) of the Appendix allows us to assert that the vectors $\mathbf{b}_3(s_0), \ldots, \mathbf{b}_n(s_0)$ can be extended throughout $s \in [s_1, s_2]$ so that the set

$$\mathfrak{F}(s) = \{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}_3(s), \ldots, \mathbf{b}_n(s)\}$$

constitutes a smoothly differentiable orthonormal basis for $\mathbb{R}^n$ throughout the range of s. We define $\mathbf{a_1}(s) = \mathbf{t}(s)$ and $\mathbf{a_2}(s) = \mathbf{n}(s)$, and note that $\dot{\mathbf{b}}_j \cdot \mathbf{b}_j = 0$ for all $j = 1, \ldots, n$.

Since $\mathfrak{F}(s)$ is a basis for $\mathbb{R}^n$ for each s in the specified interval, and j $\mathbf{b}_j(s)$ is differentiable for every j, we may write,

$$\dot{\mathbf{b}}_j(s) = \sum_{k \neq j} \alpha_k(s)\mathbf{b}_k(s), \quad j = 1, \ldots, n$$

where the αs are smooth scalar functions analogous to curvature and torsion in the 3D case. In fact, for the case $j = 1$,

$$\dot{\mathbf{b}}_1 = \dot{\mathbf{t}} = \kappa(s)\mathbf{n} = \kappa(s)\mathbf{b}_2$$

The crucial fact here is that for $j \geq 3$, $\dot{\mathbf{b}}_j \cdot \mathbf{t} = 0$, as is easily proved:

$$\frac{d}{ds}(\mathbf{b}_j \cdot \mathbf{t}) = \dot{\mathbf{b}}_j \cdot \mathbf{t} + \mathbf{b}_j \cdot \dot{\mathbf{t}} = \dot{\mathbf{b}}_j \cdot \mathbf{t} + \kappa\mathbf{b}_j \cdot \mathbf{n}$$

For $j \geq 3$, $\mathbf{b}_j \cdot \mathbf{t} = 0 = \mathbf{b}_j \cdot \mathbf{n}$, therefore, as asserted,

$$\dot{\mathbf{b}}_j \cdot \mathbf{t} = 0, \quad j \geq 3 \tag{12}$$

and for $j = 1, 2$,

$$\dot{\mathbf{t}} \cdot t = 0, \quad \dot{\mathbf{n}} \cdot \mathbf{t} = -\kappa \tag{13}$$

These facts reveal that $\mathbf{t}$ and $\mathbf{n}$ play a special role, hence by implication the $(n-2)$-dimensional binormal co-space will also play a special role.

Similar to what we did in the two-dimensional case, it is convenient to regard a point X in $\mathbb{R}^n$ as parameterized by arc length s and coordinates $(\nu, \beta_3, \ldots, \beta_n)$ in the rectangular $(n-1)$-dimensional $\{\mathbf{n}, \mathbf{b}_3, \ldots \mathbf{b}_n\}$ frame,

$$X(s, \nu, \beta_3, \ldots \beta_n) = \mathbf{x}\,(s) + \nu\,\mathbf{n}\,(s) + \sum_{j \geq 3} \beta_j \mathbf{b}_j\,(s) \tag{14}$$

We define a tendril in terms of a mapping $\mathcal{M}$ from the parameter domain $(s, \nu, \beta_3, \ldots, \beta_n) \in \mathbf{P}$ to its n-dimensional model $\mathbf{T}$ in $\mathbb{R}^n$:
(15)
$$\mathcal{M} \to \mathbf{T} = \mathcal{M}(\mathbf{P}) : (s, \nu, \beta_3, \ldots \beta_n) \mapsto X = \mathbf{x}(s) + \nu \mathbf{n}(s) + \sum_{j \geq 3} \beta_j \mathbf{b}_j(s)$$

As in the two-dimensional case, $\mathbf{x}$ is called the centerline of the tendril.

We proceed in analogy with the two-dimensional case. We are interested in the volume of the tendril in the intersection with normal planes for $s \in [s_1, s_2]$. So let $\mathbf{P}[s_1, s_2]$ and $\mathbf{T}[s_1, s_2]$ be the corresponding domain restrictions for the variables defining the tendril. Therefore, the specified volume is given by

$$V[s_1, s_2] = \int_{T[s_1, s_2]} dx_1 \cdots dx_n = \int_{P[s_1, s_2]} J_{\mathcal{M}} d\nu d\beta_3 \cdots d\beta_n ds$$

where $J_{\mathcal{M}}$ is the Jacobian of $\mathcal{M}$:

$$J_{\mathcal{M}} = \left| \det \begin{bmatrix} \frac{\partial X}{\partial s} & \frac{\partial X}{\partial \nu} & \frac{\partial X}{\partial \beta_3} & \cdots & \frac{\partial X}{\partial \beta_n} \end{bmatrix}^t \right|$$
$$= \det \left| (M) \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \vdots \\ \mathbf{b}_n \end{pmatrix} \right| = 1 - \nu\kappa$$

In the preceding equation, M is an $n \times n$ matrix. The justification for the preceding equation follows. The determinant of the second matrix on the preceding line of equations is equal to 1 because the rows are a set of orthonormal vectors. The matrix M is analogous to the matrix in the second line of Equation (18) defining the Jacobian used in the proof of Theorem 2 above. Because of Equations (12) and (13), and inspection of Equation (14), the first entry of the first column of M is $1 - \nu\kappa$ and all the other diagonal entries are 1. Moreover, as is also clear from Equation (14), all subdiagonal entries are 0.[8]

It follows that the volume of the tendril in any admissible deformed position is given by the integral

(16)
$$V[s_1, s_2] = \int_{P[s_1, s_2]} (1 - \nu\kappa) \, d\nu d\beta_3 \cdots d\beta_n ds$$
$$= \int_{s_1}^{s_2} \left(\int_{\mathbf{P}[s]} d\nu d\beta_3 \cdots d\beta_n \right) ds - \int_{s_1}^{s_2} \kappa \left(\int_{\mathbf{P}[s]} \nu d\nu d\beta_3 \cdots d\beta_n \right) ds$$

[8]The restriction to non-overlapping normal sections requires that the Jacobian $J_{\mathcal{M}} = 1 - \nu\kappa$ not vanish, just as in the three-dimensional case.

where $\mathbf{P}[s]$ denotes the normal section of the tendril at s. The last (inner) integral vanishes because the coordinates (ν, β) are measured from the centroid $\mathbf{x}(s)$ of $\mathbf{P}[s]$. Therefore, finally,

$$V[s_1, s_2] = \int_{s_1}^{s_2} \left(\int_{\mathbf{P}[s]} d\nu d\beta_3 \cdots d\beta_n \right) ds = \int_{s_1}^{s_2} A(s) ds \tag{17}$$

where $A(s)$ denotes the $(n-1)$-dimensional volume of $\mathbf{P}[s]$, the cross-section of the tendril at s. This result is independent of the specific manner of twisting of $\mathbf{x}$ in $\mathbb{R}^n$.

Note that the last inner integral of Equation (16) shows that we can weaken the hypothesis that the centerline pass through the centroid of the cross-section; we need only require that the first moment about the binormal co-space, i.e., the vector space determined by $(\mathbf{b}_3, \cdots \mathbf{b}_n)$, vanish for all $s \in (s_1, s_2)$. In this sense, Equation 17 extends Cavilieri's principle in $\mathbb{R}^n$.

Theorem 4: *Volume Invariance in* $\mathbb{R}^n$. Admissible deformations of a tendril in $\mathbb{R}^n$ preserve volume. The centroid of the normal section of the tendril need not lie on the centerline; it is sufficient that the centroid of the normal section at $\mathbf{x}(s)$ lie within the $(n-2)$-dimensional binormal co-space at $\mathbf{x}(s)$, i.e., the orthogonal complement of the plane determined by $\{\mathbf{t}(s), \mathbf{n}(s)\}$. The volume is given by Eq. (17), where $A(s)$ is the $(n-1)$-dimensional volume of the normal section of the tendril at $\mathbf{x}(s)$.

7. Conclusion

The theory in this paper arose in a study of the biological phenomenon of tendril coiling. An earlier and independent stream of thought initiated in the 1930s by Weyl and Hotelling used "tubes" to investigate properties of manifolds. As background, some basic properties of tubes in $\mathbb{R}^3$ are discussed here, and relationships with the tendril model are noted. As a general reference see *Tubes* by Gray [11]; additional selected references are given at the end of this section.

I quote Gray's definition of a tube [11, P 32]:

Definition: Tube. Let P be a topolgically embedded submanifold (possibly with boundary) in a Riemannian manifold M. Then a **tube** $T(P, r)$ of radius $r \geq 0$ about P is the set

$$T(P, r) = \{m \in M | \text{there exists a geodesic } \xi \text{ of length}$$

$$L(\xi) \leq r \text{ from } m \text{ meeting } P \text{ orthogonally}\}$$

A tube can be thought of as a thinly layered volume of uniform thickness surrounding a submanifold; the tube has the same number of dimensions as the ambient manifold. The layer must be thin enough to avoid self-intersection. A simple example is provided by a curve of finite length in $\mathbb{R}^n$ surrounded by a continuum of $(n-1)$–dimensional spheres of a fixed small radius centered on the curve and orthogonal to the curve—note that the tube does not have spherical bulbs at the tips. This is equivalent to a tendril with uniform spherical normal sections. Hotelling proved that, up to a small error dependent on characteristics of the manifold and the fourth order of the radius of the sphere, r^4, the volume of such a tube is independent of the way in which the tube is embedded in the manifold, and he gave an estimate for the volume.

The tube surrounding a curve in $\mathbb{R}^n$ is a special case of a Hotelling tube. We can use our Theorem 4 to determine the exact volume of a Hotelling tube embedded in $\mathbb{R}^n$. According to Equation (17) we only need to determine $A(s)$ and multiply it by the length of the curve, say, l. In this case, $A(s)$ is the volume of an $(n-1)$–dimensional sphere of fixed radius, say r. Since the latter is commonly known from calculus books [1], we have immediately Hotelling's Tube Formula [11, p205] for the restricted case of a one-dimensinal tube in n–dimensional Euclidean space:

$$V_n(r) = \frac{\pi^{(n-1)/2} r^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} l$$

This formula holds independently of curvature and torsion of the curve, provided that the curve does not intersect itself, and r is small enough to avoid self-intersection of the tube.

Additional simple examples are obtained by tubes covering a compact surface in $\mathbb{R}^3$. In this case, a tube will consist of line segments of fixed small length δ projecting orthogonally from both sides of the surface. We will show that volume invariance for a tube of fixed depth cannot apply in general to surfaces, i.e., to surfaces of fixed area that can be warped into arbitrary non-self-intersecting shapes. But we shall see that because of the Gauss-Bonnet theorem, volume invariance does apply to an important restricted class of surfaces.

Consider a suitably smooth surface $\mathbf{S}$ parameterized by,

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in D$$

for some domain D. Let $\mathbf{n}$ be the unit normal to the surface:

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$$

The surface S can be coated on both sides with a tubular volume of total thickness r by

$$T(u,v,t) = \mathbf{r}(u,v) + t\mathbf{n}(u,v), \quad (u,v) \in D, t \in [-r/2, r/2]$$

where r is taken small enough to ensure that the Jacobian calculated below never changes sign, to avoid self-intersection. The volume of the tube is given by

$$V = \iint dxdydz = \iint J_T dudvdt \quad (u,v) \in D, t \in [-r/2, r/2]$$

The Jacobian J_T can be calculated as:

$$J_T = \det \begin{vmatrix} T_u \\ T_v \\ T_t \end{vmatrix} = \begin{pmatrix} \mathbf{r}_u + t\mathbf{n}_u \\ \mathbf{r}_v + t\mathbf{n}_v \\ \mathbf{n} \end{pmatrix} = \tag{18}$$

$$\det \begin{vmatrix} (1+\alpha t)\mathbf{r}_u + \beta t \mathbf{r}_v \\ a t \mathbf{r}_u + (1+bt)\mathbf{r}_v \\ \mathbf{n} \end{vmatrix} = \det \begin{vmatrix} (1+\alpha t) & \beta t & 0 \\ at & (1+bt) & 0 \\ 0 & 0 & 1 \end{vmatrix} \det \begin{vmatrix} \mathbf{r}_u \\ \mathbf{r}_v \\ \mathbf{n} \end{vmatrix}$$

Where, by Weingarten's formulas for $\mathbf{n}_u$ and $\mathbf{n}_v$ [21],

$$\alpha = \frac{fF - eG}{EG - F^2}, \quad \beta = \frac{eF - fE}{EG - F^2}$$

$$a = \frac{gF - fG}{EG - F^2}, \quad b = \frac{fF - gE}{EG - F^2}$$

The variables e, g, f and E, G, F are coefficients in the first and second fundamental forms for S, respectively.

We will use the fact that the last determinant of Equation (18) is equal to $\sqrt{EG - F^2}$ and that the area element is $dA = \sqrt{EG - F^2}\, dudv$. But first expand the penultimate determinant, call it D, to obtain,

$$D(u,v,t) = 1 + (\alpha + b)t + (\alpha b - a\beta)t^2 = 1 - 2Ht + Kt^2 \tag{19}$$

where, remarkably, we find H the mean curvature at $r(u,v)$ and K the Gaussian curvature:

$$H = \frac{1}{2} \cdot \frac{Eg - 2fF + eG}{EG - F^2}$$

and

$$K = \frac{eg - f^2}{EG - F^2}$$

Because t is an odd function, our formula for the volume is just,

$$V = \iint D \, dA = \tag{20}$$

$$\int_S \int_{t=-r/2}^{r/2} \left\{1 + Kt^2\right\} \, dtdA = rA + \frac{r^3}{12} \int_S K \, dA$$

On the latter line of equations, A is the area of the surface S, and the last integral is the *integral curvature* of the surface S. The last integral is evaluated by the Gauss-Bonnet Theorem [18]: If S is a compact orientable smooth surface (without boundary)

$$\iint_S K \, dA = 2\pi\chi(S)$$

where $\chi(S)$ is the Euler characteristic of S and dA is the area element.

Thus our formula for V in the case of such a compact surface simplifies to,

$$V = rA + \frac{\pi r^3}{6}\chi(s)$$

We now have the remarkable result that an area preserving diffeomorphism of such a compact surface S leaves the tube volume unchanged; this is a form of volume invariance under allowable deformations of a restricted class of surfaces. But the more general formula for V above shows that we cannot deduce the same result for S an arbitrary surface patch.

As examples of invariance, let us just note that for a sphere S, $\chi(S) = 2$. So any area-preserving diffeormorphism of a sphere will result in an invariant volume of

$$V = rA + \frac{\pi r^3}{3}$$

for a tube of total thickness r. Similarly, for S a torus, $\chi(s) = 0$, and therefore any area-preserving diffeormorphism of a torus will result in an invariant volume of

$$V = rA$$

for a tube of total thickness r. Comparable results for tubes covering curves and surfaces in $\mathbb{R}^3$ are derived differently in the first few pages of Gray [11].

To test whether volume invariance can be generalized by allowing the depth of a tube to vary, observe that the formula for the volume of a tube given by Equation (20), in which the uniform depth was assumed to be $r/2$, can be modified to give volume for a tube in which the depth varies with position on the surface. This gives us,

$$V = \iint D \, dA = \tag{21}$$

$$\int_S \int_{\rho_l(u,v)}^{\rho_u(u,v)} \left\{1 - 2Ht + Kt^2\right\} \, dtdA$$

Where ρ_l and ρ_u are the "lower" and "upper" spherical radii, which in this general case are allowed to vary with points on S. In general the multiple integral cannot be factored as in Equation (20), showing that volume invariance cannot hold in general.

Because the machinery lies at hand, we discuss the famous definition of curvature given by Gauss and follow with one further observation about tubes surrounding a surface in $\mathbb{R}^3$. Let S be a smooth surface parameterized by $\mathbf{r}(u,v)$. Consider a unit normal $\mathbf{n}(u,v)$ to S as it moves around a neighborhood of a point $P_0 = \mathbf{r}(u_0, v_0)$ on S determined by $(u,v) \in U_0$, where U_0 is a small open set in $\mathbb{R}^2$ containing (u_0, v_0) in the parameter domain for S. We assume that S is well-oriented by the choice of $\mathbf{n}$. The *Gauss map* sets up a correspondence between points on a smooth surface S and points on a unit sphere. We can implement the Gauss map for S by $\mathbf{R}(u,v) = \mathbf{n}(u,v), (u,v) \in U_0$, and evaluate the corresponding area on the sphere,

$$\iint_{U_0} ||\mathbf{R}_u \times \mathbf{R}_v||\, dudv = \iint_{U_0} ||\mathbf{n}_u \times \mathbf{n}_v||\, dudv$$

As we know from above,

$$\mathbf{n}_u = \alpha \mathbf{r}_u + \beta \mathbf{r}_v, \quad \mathbf{n}_v = a\mathbf{r}_u + b\mathbf{r}_v \tag{22}$$

where the coefficients α, β, a, b are given by Weingarten's formulas. And from our previous calculations, we infer,

$$\mathbf{n}_u \times \mathbf{n}_v = (\alpha b - a\beta)(\mathbf{r}_u \times \mathbf{r}_v) = K(\mathbf{r}_u \times \mathbf{r}_v)$$

where K is a quantity previously identified as Gaussian curvature. Therefore, the area of the corresponding spherical surface is

$$\iint_{U_0} ||\mathbf{n}_u \times \mathbf{n}_v||\, dudv = \iint_{U_0} K||\mathbf{r}_u \times \mathbf{r}_v||\, dudv = \iint_{U_0} K\, dA$$

where dA is a surface element of S in the neighborhood of P_0. Thus we see that Gauss' definition of curvature—the relative density of spherical area to surface area on S at an arbitrary point P_0—is identical to K at P_0, validating the identification.

But it is natural to ask now, exactly how does this last formula relate to the area above a patch of the surface of S? To answer, note that in analogy with the way we parameterized tube volume around S, we can parameterize the surface at a given distance t measured along $\mathbf{n}(u,v)$, the unit normal vector to S selected above:

$$\mathbf{s}(u,v) = \mathbf{r}(u,v) + t\mathbf{n}; \quad t \in [-r/2, r/2] \tag{23}$$

The area for this surface is,

$$A(t) = \iint_S ||\mathbf{s}_u \times \mathbf{s}_v||\, dudv = \iint_S ||(\mathbf{r}_u + t\mathbf{n}_u) \times (\mathbf{r}_v + t\mathbf{n}_v)||\, dudv$$

Using again Equations (19) and (22), we obtain,

$$\begin{aligned}(24)\quad (\mathbf{r}_u + t\mathbf{n}_u) \times (\mathbf{r}_v + t\mathbf{n}_v) &= [1 + (\alpha + b)t + (\alpha b - a\beta)t^2](\mathbf{r}_u \times \mathbf{r}_v) \\ &= \left(1 - 2Ht + Kt^2\right)(\mathbf{r}_u \times \mathbf{r}_v)\end{aligned}$$

So we have,

$$\begin{aligned}A(t) &= \iint_S [1 + (\alpha + b)t + (\alpha b - a\beta)t^2]||\mathbf{r}_u \times \mathbf{r}_v||\, dudv \\ &= \iint_S \left(1 - 2Ht + Kt^2\right)\, dA\end{aligned}$$

It is not altogether surprising that the integrand for this surface area is identical to the integrand for the volume given by Equation (19). This is because Equation (24) shows that the unit normal $\mathbf{n}$ at $\mathbf{r}(u, v)$ lies on the line normal to S at $\mathbf{s}(u, v)$.[9] So we can think of the volume of a tube around a surface in $\mathbb{R}^3$ as thin layers of curved laminae or *rinds* of uniform thickness caught between a succession of surfaces obtained by slight increments of t—each increment being the thickness of the rind—and the volume of the rind being approximately equal to its thickness times the area of the underlying surface. Thus, for the volume caught between surfaces at t_1 and t_2,

$$V[t_1, t_2] = \int_{t_1}^{t_2} A(t)\, dt$$

And so it is that the volume for a tube of total thickness r can be calculated by

$$\begin{aligned}V[-r/2, r/2] &= \int_{-r/2}^{r/2} A(t)\, dt = \int_{-r/2}^{r/2} \left(1 - 2Ht + Kt^2\right)\, dA \\ &= rA + \frac{r^3}{12}\iint_S K\, dA\end{aligned}$$

the same expression of Equation (19) we derived using the Jacobian of Equation (18) in a formula for the volume of the tube.

Before concluding, I would like to mention a feature of the mean curvature brought out by the negative sign in the factor of t in Equation (24) and in the subsequent formula for $A(t)$: At points of S where H

[9]So the family of hypersurfaces thus defined and indexed by t in Equation (23) constitute a natural foliation of the tubular volume surrounding S. This characteristic of tubes in general is mentioned by Gray [11, P 33].

is positive, a positive value of t yields a smaller surface density than does $-t$. This is because, if the normal to the surface S defined by $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ points outward from a concavity in S, then H is negative. In that case, the map given by (23) for a positive value of t defines a portion of surface lying *within* the concavity, and hence it has less area than the corresponding portion of S. And for negative values of t the map of Equation (23) defines a larger portion of area lying "above" the concavity. The opposite situation prevails if the normal to the S defined by $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ points into a concavity of S, in which case H is positive. We may say that $A(t)$ demonstrates *fan-in* for the surface generated by Equation (23) for a portion of the surface within a concavity of S, and *fan-out* for a portion outside a concavity.

I conclude by mentioning that tubes are discussed in many books about differential geometry and manifolds, either in cameo or in applications. Here I make an arbitrary selection for reference. Foremost, Gray [11] is a treatise devoted to the theory of tubes and a wide range of applications with historical notes. Do Carmo [9] uses a tube surrounding a simple closed curve in $\mathbb{R}^3$, and the Gauss-Bonnet theorem applied to the surface of the tube, to prove Fenchel's theorem that a closed curve in $\mathbb{R}^3$ has total curvature $\geq 2\pi$, and he extends the method to prove the Fary-Milnor theorem that a simple knotted closed curve has total curvature $\geq 4\pi$. Dubrovin, Fomenko and Novikov [2] apply tubular neighborhoods to prove the existence of Morse functions on compact submanifolds. Further discussions of tubes can be found in Hirsch [12], Milnor and Stasheff [17], and Spivak [20].

For relevant background in analysis, differential topology and manifold theory see Boothby [3], Th. Bröcker & K. Jänich [13], Knapp [14], McCleary [15], and Milnor [16].

8. Appendix 1: Moving Frame

In proving the Volume Invariance Theorem for n dimensions, I used the fact that it is possible to choose a smoothly varying set of orthonormal basis vectors for the othogonal complement of the span of the vectors $\{\mathbf{t}(s), \mathbf{n}(s)\}$, which vary smoothly throughout some nonempty closed interval I. This fact is validated by the following Lemma (1).

We adapt the Gram-Schmidt orthogonalization process. Given $n > 2$ and an $n \times n$ orthogonal matrix, the first $k \geq 2$ columns of which vary smoothly throughout I (say, $C^1[I]$) and are pairwise orthogonal and of unit length for all $s \in I$. Denote by $\mathbf{M}_0$ the matrix at $s = s_0$. Then it is possible to extend the remaining $n - k$ columns smoothly wrt s so that the resulting matrix $\mathbf{M}(s)$ is orthogonal and varies smoothly ($C^1[I_0]$) as a function of $s \in I_0$ for some open interval I_0 around s_0. Note that in the case $n = 3$, this process will yield $\mathbf{b}(s)$ or its negative.

Use Gram-Schmidt inductively. Denote the j-th column of the matrix by $\mathbf{V}_j$. For $j \leq k$, express the j-th column as $\mathbf{V}_j(s)$, which by assumption is a smoothly varying function of s. Let

$$\mathbf{W}(s) = \mathbf{V}_{k+1} - \sum_{j=1}^{k} \left(\mathbf{V}_{k+1} \cdot \mathbf{V}_j(s)\right) \mathbf{V}_j(s) \tag{25}$$

and define

$$\mathbf{V}_{k+1}(s) = \frac{\mathbf{W}(s)}{||\mathbf{W}(s)||} \tag{26}$$

Because all of the factors on the right-hand side of Equation (25) vary smoothly, so does $\mathbf{W}(s)$. Moreover, the set $\{\mathbf{V}_{k+1}, \mathbf{V}_1(s_0), \ldots, \mathbf{V}_k(s_0)\}$ is orthonormal, and for sufficiently small variations of s around s_0, $\mathbf{W}(s)$ is bounded away from 0. Thus, for these small variations of s, Equation (26) defines a smoothly varying function such that the set of vectors $\{\mathbf{V}_{k+1}(s), \mathbf{V}_1(s), \ldots, \mathbf{V}_k(s)\}$ is orthonormal.

This argument may be extended inductively so that finally, for s within some open interval containing s_0, we have a smoothly varying orthogonal matrix $\mathbf{M}(s)$ such that $\mathbf{M}(s_0) = \mathbf{M}_0$. A disadvantage of this line of argument is that we have no control over the size of the resulting interval around s_0.

The difficulty arises because the expression for $\mathbf{W}(s)$ in Equation (25) may vanish for some value or values of s invalidating Equation (26). The fact that this cannot happen for sufficiently small values of s suggests that we could define $\mathbf{W}$ *locally*, i.e., in terms of a differential equation.

So we start backwards by assuming that there exists a unit vector valued function $\mathbf{W}(s)$ that is everywhere differentiable and orthogonal to the class C^3 vector valued function $\mathbf{v}(s)$. Then, using Gram-Schmidt, we could write

$$\mathbf{W}(s+\delta) = \mathbf{W}(s) - (\mathbf{W}(s)\cdot\mathbf{V}(s+\delta))\,\mathbf{V}(s+\delta)$$

Using the assumption that $\mathbf{W}(s)\cdot\mathbf{V}(s) = 0$, we can rewrite the foregoing equation as

$$\mathbf{W}(s+\delta) - \mathbf{W}(s) = -\left[\mathbf{W}(s)\cdot(\mathbf{V}(s+\delta) - \mathbf{V}(s))\right]\mathbf{V}(s+\delta)$$

Dividing by δ and passing to the limit, we get the linear system of differential equations,

$$\dot{\mathbf{W}}(s) = \mathbf{W}(s) - \left(\mathbf{W}(s)\cdot\dot{\mathbf{V}(s)}\right)\mathbf{V}(s)$$

This system is solvable by the *Cauchy–Peano Existence Theorem* because on the right-hand side of the differential equation the coefficients of the coordinates of $\mathbf{W}$ are continuous functions of the independent variable s [6, p. 6]. Note that the theorem only asserts existence not uniqueness of a solution; this is sufficient for us. This line of reasoning suggests the following.

Lemma 1. (**Global Moving Frame**) Given an $n \times n$ orthogonal matrix, $\mathbf{M}(s)$, the first k columns of which are of class $C^3[I]$ wrt to an independent variable s on a closed interval I, such that the first k columns are pairwise orthogonal and of unit length throughout I, and the remaining columns are constants. Given a fixed s_0 in I, let $\mathbf{M}_0 = \mathbf{M}(s_0)$. Then it is possible to extend the remaining $n-k$ columns smoothly wrt s so that the resulting matrix $\mathbf{M}^*(s)$ is orthogonal and varies smoothly ($C^1[I]$) as a function of s for all s in I.

Notation. We use $\mathbf{V}_i(s), i = 1, \ldots, k$ to denote the first k columns of $\mathbf{M}(s)$. The remaining columns are constants, and we denote them by $\mathbf{V}_i, i = k+1, \ldots, n$.

Comment. The proof is by mathematical induction. We begin with $\mathbf{V}_1(s) = \mathbf{t}(s)$ and $\mathbf{V}_2(s) = \mathbf{n}(s)$, the mutually orthogonal unit tangent vector and principal normal vector to a given space curve, for which the interval of definition is $s \in I$ for some non-empty closed interval I. We begin the induction with $k = 3$. The Lemma asserts existence but not uniqueness of $\mathbf{M}^*(s)$.

Proof. The argument is inductive on k. We define a vector valued function $\mathbf{W}(s)$ by the system of linear differential equations,

$$-\dot{\mathbf{W}} = \sum_{j=1}^{k} \left(\mathbf{W} \cdot \dot{\mathbf{V}}_j \right) \mathbf{V}_j, \quad \mathbf{W}(s_0) = \mathbf{V}_{k+1} \tag{27}$$

This system has a continuously differentiable solution (not necessarily unique) throughout I because the coefficients of $\mathbf{W}$ on the right-hand side of Equation (27) are continuous (See [6, p. 6]). We show that $\mathbf{W}(s)$ is a unit vector orthogonal to all $\mathbf{V}_i(s)$ for $i = 1, \ldots, k$ and all s in I:

$$-\dot{\mathbf{W}} \cdot \mathbf{V}_i = \sum_{j=1}^{k} \left(\mathbf{W} \cdot \dot{\mathbf{V}}_j \right) \mathbf{V}_j \cdot \mathbf{V}_i = \mathbf{W} \cdot \dot{\mathbf{V}}_i$$

The latter equation implies that

$$\frac{d}{ds} \left(\mathbf{W} \cdot \mathbf{V}_i \right) = 0$$

Therefore, $(\mathbf{W} \cdot \mathbf{V}_i)$ must be constant. Since by choice $\mathbf{W}(s_0)$ is orthogonal to $\mathbf{V}_i(s_0)$, we find that,

$$(\mathbf{W}(s) \cdot \mathbf{V}_i(s)) = 0, \quad s \in I$$

It follows that $\mathbf{W}$ is orthogonal to $\mathbf{V}_i$ for every $i = 1, \ldots, k$ throughout I. We can now see that $\mathbf{W}$ is a unit vector by multiplying Equation (27) by $\mathbf{W}$ to obtain,

$$-\frac{1}{2} \frac{d}{ds} (\mathbf{W} \cdot \mathbf{W}) = -\dot{\mathbf{W}} \cdot \mathbf{W} = 0$$

Thus, $|\mathbf{W}|$ is constant, and for $s = s_0$ it equals 1.

Setting $\mathbf{V}_{k+1}(s)$ equal to $\mathbf{W}(s)$ for s in I, we obtain a set of differentiable vectors $\{\mathbf{V}_1(s), \ldots, \mathbf{V}_{k+1}(s)\}$ that is orthonormal for all s in I. This argument may be extended inductively to $k = n$ so that finally we arrive at a continuously differentiable orthogonal matrix $\mathbf{M}^\star(s)$ such that $\mathbf{M}^\star(s_0) = \mathbf{M}(s_0)$. □

9. Appendix 2: Pappus I not Extensible to higher dimensions

I have generalized Pappus II to curves in three dimensions and higher. It is natural to ask whether Pappus I can also be generalized. To show that this cannot be done, it is sufficient to examine the case of a fixed curve within the $\mathbf{n}-\mathbf{b}$ plane of the Frenet frame as it moves along a C^3 class curve $\mathbf{x}$.

We adapt Equation (5) to define a surface in terms of arclength s, and a function defined within the $\mathbf{n}-\mathbf{b}$ plane, $f:\beta \mapsto f(\beta)$, where β is a coordinate value along the $\mathbf{b}$ axis, and $f(\beta)$ is a coordinate value along the $\mathbf{n}$ axis. We consider the restricted case in which the function f does not vary with $s \in [s_1, s_2]$. This gives a curve of fixed length and a surface of constant normal cross section, while allowing a free choice for κ and τ:

$$\mathbf{L}\,(s,\beta) = \mathbf{x}(s) + f(\beta)\mathbf{n}(s) + \beta\mathbf{b}(s) \tag{28}$$

We investigate whether the resulting surface area depends on κ and τ. For calculating surface area represented parametrically, the expression of interest is [1],

$$\mathbf{L}_s \times \mathbf{L}_\beta$$

$$= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (1-\kappa f(\beta) & -\tau\beta & \tau f(\beta) \\ 0 & f'(\beta) & 1 \end{vmatrix} \tag{29}$$

$$= -f'(\beta)\det \begin{vmatrix} \mathbf{i} & \mathbf{k} \\ (1-\kappa f(\beta)) & \tau f(\beta) \end{vmatrix} \tag{30}$$

$$+\det \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ (1-\kappa f(\beta)) & -\tau\beta \end{vmatrix} \tag{31}$$

$$= -f'\left[\tau f\mathbf{i} - (1-\kappa f)\mathbf{k}\right] - \left[\tau\beta\mathbf{i} + (1-\kappa f)\mathbf{j}\right] \tag{32}$$

$$= -\tau(ff'+\beta)\mathbf{i} - (1-\kappa f)\,\mathbf{j} + f'(1-\kappa f)\,\mathbf{k}$$

Therefore,

$$|\mathbf{L}_s \times \mathbf{L}_\theta| = \sqrt{\tau^2(ff'+\beta)^2 + (1-\kappa f)^2(1+f'^2)}$$

And the surface area swept out by the graph $\{\nu \in [\nu_1,\nu_2] : (\nu, f(\nu))\}$ as the Frenet frame moves from s_1 to s_2 is given by the integral over that domain:

$$S[s_1, s_2; \nu_1, \nu_2] = \iint \sqrt{\tau^2(ff' + \beta)^2 + (1 - \kappa f)^2(1 + f'^2)}\, ds d\nu$$

Clearly the integral depends on the choice of τ and κ, so we do not obtain a generalization of Pappus I. That the integral actually yields the surface area S asserted by Pappus I for a circular curve $\mathbf{x}$ of radius R follows by setting $\tau = 0$, $\kappa = 1/R$, and $(s_1, s_2) = (0, 2\pi R)$:

$$S = 2\pi R \int_{\nu_1}^{\nu_2} (1 - \kappa f)\sqrt{1 + f'^2}\, d\nu = 2\pi \int_{\nu_1}^{\nu_2} (R - f)\sqrt{1 + f'^2}\, d\nu$$

That this is the First Theorem of Pappus follows from observing that the length l of the curve of the specified graph is given by

$$l = \int_{\nu_1}^{\nu_2} \sqrt{1 + f'^2}\, d\nu$$

The $\mathbf{n}$–coordinate $\bar{f}$ of the centroid of the specified graph is given by,

$$l\bar{f} = \int_{\nu_1}^{\nu_2} f\sqrt{1 + f'^2}\, d\nu$$

and the circumference of a circle of radius $R - \bar{f}$ through which the centroid is rotated is $2\pi(R - \bar{f})$. Pappus I asserts that

$$S = 2\pi(R - \bar{f})l = 2\pi \int_{\nu_1}^{\nu_2} (R - f)\sqrt{1 + f'^2}\, d\nu$$

the integral displayed above.

References

1. Tom M. Apostol, *Calculus*, 2 ed., vol. 2, John Wiley & Sons, Inc., New York, 1969, (Apostol discusses the two Pappus Theorems treated in this article).
2. S. P. Novikov B. A. Dubrovin, A. T. Fomenko, *Modern geometry—methods and applications*, vol. 2, Springer-Verlag, New York, 1985.
3. William M. Boothby, *An introduction to manifolds and riemannian geometry*, 2 ed., Academic Press, Inc., Orlando, 1986.
4. R. Creighton Buck, *Advanced calculus*, 3 ed., Waveland Press, Inc., Long Grove, Ill, 1978.
5. Jr. C. H. Edwards, *Advanced calculus of several variables*, Dover Publications, Inc., New York, 1973.
6. Earl A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book company, Inc., New York, 1955.
7. Richard Courant and Fritz John, *Introduction to calculus and analysis*, vol. 2, John Wiley & Sons, Inc., New York, 1974, Written wit the assitance of Albert A. Blank and Alan Solomon.
8. Charles Darwin, *On the movements and habits of climbing plants*, NYU Press, 1989,
(Originally published on September 1, 1865 as a 118 page treatise by the Linnean Society, it was republished with an appendix and corrections in 1882; it has been published in a scholarly edition of the complete scientific writings of Darwin by NYU Press and is available elsewhere in inexpensive book format.).
9. Manfredo P. Do_Carmo, *Differential geometry of curves and surfaces*, Prentice Hall, Englewood Cliffs, 1976.
10. Alan Goriely, Professor of Mathematics at the University of Arizona, his web features articles applying continuum mechanics to elastic growth in vines and tendrils, ⟨`http://math.arizona.edu/ goriely/`⟩.
11. Alfred Gray, *Tubes*, 2 ed., Birkhäuser Verlag, Boston, 2004.
12. Morris W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976.
13. Th. Bröcker & K. Jänich, *Introduction to differential topology*, Cambridge University Press, Cambridge, 1982.
14. Anthony W. Knapp, *Advanced real analysis*, Birkhäuser, Boston, 2005.
15. John McCleary, *Geometry from a differentiable viewpoint*, Cambridge University Press, Cambridge, 1994.
16. John W. Milnor, *Topology from the differentiable viewpoint*, Princeton University Press, Princeton, 1997, Revised edition.
17. John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, 1974.
18. Barrett O'Neill, *Elementary differential geometry*, 2 ed., Academic Press, San Diego, 1997.
19. Michael R. Raugh, *Some geometry problems suggested by the shapes of tendrils*, Ph.D. thesis, Stanford University, 1978.
20. Michael Spivak, *A comprehensive introdution to differential geometry*, vol. 3, Publish or Perish , Inc., Berkeley, 1979.
21. Dirk J. Struik, *Lectures on classical differential geometry*, 2 ed., Dover Publication, New York, 1988.

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