Remembering Ray's Lecture: e *is Transcendental*

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The LACC High School Math Contest

Los Angeles City College has offered the High School Math Contest almost every year for nearly 60 years.

<www.lacitycollege.edu/academic/departments/mathdept/aboutcontest.html>

The contest is open to all students in Southern California.

The talk this year was about π and a proof sketch that π is irrational. I invited the students to fill in the details later.

This online companion to that talk sketches a proof that *e* is transcendental—with the same challenge!

Ray Redheffer was a Professor of Mathematics at UCLA 1950–1991.

I attended his lecture about *e* at UCLA in 1955.

I recalled the core ideas of the proof, and so can you.

I'll present a proof sketch for you to fill in.

It's a nice rigorous exercise in basic calculus.

Work it out, then show it to your class!

Ray was physically fit into his eighties.

Ray died of cancer in 2005 at age 84.

He was a terrific mathematician (over 200 publications), but much else besides.

- He encouraged students of all ages and gave popular lectures at public schools.
- He created many of the exhibits in the famous Eames Mathematica exhibition.
- He and Heddy built their own home.
- He could do a one-arm chin-up.
- He played classical music on his grand piano.
- He could recite long poems from memory.

Here is what we will prove in outline.

Theorem. e is **transcendental** (not an *algebraic number*), i.e., not a solution of an algebraic equation with integer coefficients.

In other words,

- e satisfies no algebraic equation with integer coefficients:

$$\boxed{a_0 e^n + a_1 e^{n-1} + a_2 e^{n-2} + \cdots + a_{n-1} e + a_n \neq 0}$$

- The first proof was by the French mathematician Claude Hermite (1873); our proof requiring elementary calculus is based on Hilbert's simplification.*

* See reference.

The key to the proof is an amazing integral.

You'll find that it's not hard to remember:

$$M(p) = \frac{1}{(p-1)!} \int_0^\infty e^{-t} t^{p-1} [(t-1)(t-2)\cdots(t-n)]^p dt$$

where p is a large prime that you will choose at the end.

The proof manipulates $M(p)$ using basic calculus.

That's how I remembered the proof after learning the bare outline, **and you can, too**.

Here's something you need to know.

This can be proved easily using *integration by parts*:

$$\boxed{\int_0^\infty e^{-t} t^k dt = k! \quad k \in \mathbb{N}_0} \; *$$

So, for prime integer p,

$$\frac{1}{(p-1)!} \int_0^\infty e^{-t} t^{p-1} t^k dt = \begin{Bmatrix} m(p), & k \geq 1 \\ \bar{m}(p), & k = 0 \end{Bmatrix} \qquad (1)$$

Where: $m(p)$ denotes an **integer multiple of** p, and
$\bar{m}(p)$ denotes an **integer non-multiple of** p.

* You may recognize this "factorial" integral as $\Gamma(k+1)$.

Here's the bare outline, a proof by contradiction.

We suppose there **are** integers $a_0, \ldots, a_n$ such that,

$$\boxed{a_0 e^n + a_1 e^{n-1} + a_2 e^{n-2} + \cdots + a_{n-1} e + a_n = 0}$$

Define $M = M(p)$, then split it into two parts so that:

$$M = e^{-k} M_k + e^{-k} \epsilon_k, \quad \epsilon_0 \equiv 0 \tag{2}$$

$$\boxed{\begin{gathered} a_0 M_n + a_1 M_{n-1} + a_2 M_{n-2} + \cdots + a_{n-1} M_1 + a_n M_0 + \\ a_0 \epsilon_n + a_1 \epsilon_{n-1} + a_2 \epsilon_{n-2} + \cdots + a_{n-1} \epsilon_1 = 0 \end{gathered}}$$

The crux is to choose M_k and ϵ_k to make sure the row of Ms is a non-zero integer, and the row of ϵs is very small—**the contradiction**!

Prepare to split $M(p)$ into two parts.

By definition $M = M(p)$, and from above M_0 must equal M:

$$M = \frac{1}{(p-1)!}\int_0^\infty e^{-t}t^{p-1}[(t-1)\cdots \boxed{(t-k)} \cdots (t-n)]^p dt$$

Let $\boxed{T = t - k}$, and **change variables** (for $k \in \{1, 2, \ldots, n\}$):

$$M = \frac{e^{-k}}{(p-1)!}\int_{-k}^\infty e^{-T}(T+k)^{p-1}[\cdots \boxed{T} \cdots]^p dT$$

To clarify operations, move e^{-k} to the other side:

$$e^k M = \frac{1}{(p-1)!}\int_{-k}^\infty e^{-T}(T+k)^{p-1}[\cdots \boxed{T} \cdots]^p dT$$

Now do the split.

$$e^k M = \frac{1}{(p-1)!} \int_{-k}^{\infty} e^{-T}(T+k)^{p-1}[\cdots \boxed{T} \cdots]^p dT$$

Define the split, for $k \in \{1, 2, \ldots, n\}$ by,

$$\epsilon_k = \frac{1}{(p-1)!} \int_{-k}^{0} e^{-T}(T+k)^{p-1}[\cdots \boxed{T} \cdots]^p dT$$

$$M_k = \frac{1}{(p-1)!} \int_{0}^{\infty} e^{-T}(T+k)^{p-1}[\cdots \boxed{T} \cdots]^p dT$$

The case for $k = 0$ was dealt with above. Complete the proof by showing that (1) $M_0 = \bar{m}(p)$, (2) $M_k = m(p)$, and (3) $\epsilon_k \to 0$ as $p \to \infty$. (Refer to the bare outline and Equations (1) & (2) on previous slides.)

Here's some basic analysis you might use:

- $e^t \equiv 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$ (converges for all t)
- $\frac{t^k}{k!} \to 0$ as $k \to \infty$ (because the series converges)
- $e^t > \frac{t^k}{k!}$ (for all $t > 0$ and integers $k \geq 0$)
- $\lim_{t\to\infty} e^{-t} t^k = 0$ (because $e^t t^{-k} > \frac{t^{(k+1)}}{(k+1)!} \cdot t^{-k} = \frac{t}{(k+1)!}$)
- $\frac{d}{dt} e^{-t} = -e^{-t}$
- $\int_0^\infty e^{-t} dt = -e^{-t}|_0^\infty = -(e^{-\infty} - e^{-0}) = \boxed{1}$
- Integration by parts: $\int u dv = uv - \int v du$

Here's a scheme to remember the proof:

1) Remember the amazing integral,

$$M = \frac{1}{(p-1)!} \int_0^\infty e^{-t} t^{p-1} [(t-1)(t-2)\cdots(t-n)]^p dt$$

and suppose there are integers $a_0, \ldots, a_n$ such that,

$$\boxed{a_0 e^n + a_1 e^{n-1} + a_2 e^{n-2} + \cdots + a_{n-1} e + a_n = 0}$$

2) Split M into two parts, $M = e^{-k} M_k + e^{-k} \epsilon_k$, such that,

$$a_0 M_n + a_1 M_{n-1} + a_2 M_{n-2} + \cdots + a_{n-1} M_1 + a_n M_0 + a_0 \epsilon_n + a_1 \epsilon_{n-1} + a_2 \epsilon_{n-2} + \cdots + a_{n-1} \epsilon_1 \neq 0$$

References

- Theodore W. Gamelin, “IN MEMORIAM, Raymond Redheffer, Professor of Mathematics, Emeritus, Los Angeles, 1921-2005.” (Accessible on Internet)
- Mike Raugh, “Ray Redheffer Remembered” (on my web)
- R. M. Redheffer and R. Steinberg, “Analytic proof of the Lindemann theorem”, *Pacific Journal of Mathematics*, Volume 2, Number 2 (1952), 231-242. (Accessible on Internet) [The proof shown here follows the one in this article, which also proves that π is transcendental.]
- Wikipedia entry for “*Transcendental number*”.

A challenge for the contestants:

If you know basic calculus,
try filling in the details of this proof!

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