NOTES

Reconsidering Leibniz's Analytic Solution of the Catenary Problem: The Letter to Rudolph von Bodenhausen of August 1691

Mike Raugh

January 23, 2017

Leibniz published his solution of the catenary problem as a classical ruler-and-compass construction in the June 1691 issue of Acta Eruditorum, without comment about the analysis used to derive it.[1] However, in a private letter to Rudolph Christian von Bodenhausen later in the same year he explained his analysis.[2]

Here I take up Leibniz's argument at a crucial point in the letter to show that a simple observation leads easily and much more quickly to the solution than the path followed by Leibniz. The argument up to the crucial point affords a showcase in the techniques of Leibniz's calculus, so I take advantage of the opportunity to discuss it in the Appendix.

Leibniz begins by deriving a differential equation for the catenary, which in our modern orientation of an $x-y$ coordinate system would be written as,

$$\frac{dy}{dx} = \frac{n}{a} \qquad (n = \int \sqrt{dx^2 + dy^2}), \tag{1}$$

where (x, z) represents cartesian coordinates for a point on the catenary, n is the arc length from that point to the lowest point, the fraction on the left is a ratio of differentials, and a is a constant representing unity used throughout the derivation to maintain homogeneity.[3] The equation characterizes the catenary, but to solve it n must be eliminated.

[1] Leibniz, Gottfried Wilhelm, "De linea in quam flexile se pondere curvat" in *Die Mathematischen Zeitschriftenartikel*, Chap 15, pp 115–124, (German translation and comments by Hess und Babin), Georg Olms Verlag, 2011.

I thank Professor Eberhard Knobloch for citing this work and discussing it with me at his office in Berlin. Professor Knobloch is director of the Alexander von Humboldt Research Centre of the Berlin-Brandenburg Academy of Sciences, and head of the math and technical-scientific sections of the Academy edition of the works of Gottfried Wilhelm Leibniz.

[2] Letter in German to Rudolf Christan von Bodenhausen, Wolfenbutel, 10/20 August 1691, and in the attached Latin text "Analysis problemas catenari", both printed in G. W. Leibniz, *Sämtliche Schriften und Briefe*, series III, volume 5 (2003), p. 143-155; `http://www.gwlb.de/Leibniz/Leibnizarchiv/Veroeffentlichungen/III5A.pdf`.

I thank Dr. Siegmund Probst of the Leibniz Archive at the Göttingen Academy of Science for bringing this letter to my attention, for a rush translation into English of the Latin text and for a helpful exchange of emails.

[3] The same equation was found by Johann Bernoulli in his solution of the catenary problem. See the translation by William A. Ferguson, Jr. at, `http://www.21stcenturysciencetech.com/translations/Bernoulli.pdf`. The equation is a consequence of the parallelogram law of statics and the fact that tension must act in the direction of the tangent.

With some algebraic manipulation of Equation 1 using second differentials and integration that Leibniz assumed his reader would understand, he showed that,

$$dx = \frac{a\,dy}{\sqrt{2ay + y^2}} \tag{2}$$

then, comparing this with Equation 1 and substituting $z = y + a$, concluded that $n = \sqrt{z^2 - a^2}$.[4] The reader might think that this equation for n has allowed us to determine an arc length without the use of integration, but the derivation of Equation 2 demonstrated in the Appendix does in fact require integration.

My departure from Leibniz and simplification of the argument begins here, where Leibniz makes use of the product of $z + n$ and $z - n$:

$$(z + n)(z - n) = a^2.$$

I assume, without essential loss of generality, that $a = 1$ and that for some yet unknown n we have,

$$(z + n)(z - n) = 1. \tag{3}$$

The product expressed by Equation 3 hints at a role for the exponential function $z = e^x$, since, if we have $p \cdot q = 1$, we will have logarithms for p and q that are bilaterally symmetric about the origin. In our case, we would have, $\ln(z + n) = -\ln(z - n)$. But the curve for $z = e^x$ is not bilaterally symmetric, so it cannot by itself represent a catenary. Let's try the next simplest thing: *symmetrize* the exponential by letting:[5]

[4] Leibniz's derivation of Equation 2 from Equation 1 is a good example of his algebraic use of differentials. For this, and verification that Equation 2 follows from Equation 1, see Appendix.

[5] Testing simple guesses is a good technique for exploring groundwork in a difficult problem; by good luck the simplest guess for this problem yields the solution straight away. We recognize this from familiarity with the problem, but for full verification it is necessary to proceed with the argument.

$$z = \frac{e^x + e^{-x}}{2},$$

This formula for z *is* bilaterally symmetric, so let us find an n such that Equation 3 is satisfied. The result is immediate and decisive, since, by Equation 3, $n^2 = z^2 - 1$:

$$n^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = \left(\frac{e^x - e^{-x}}{2}\right)^2.$$

We choose $n = \frac{e^x - e^{-x}}{2}$ and conclude that $dz = dy = n\,dx$. This is the differential equation that Leibniz derived to characterize the catenary, where *his* n = arc length. But is the n *we* derived arc length too? Yes, it is, as can be shown easily from the above with these definitions and results:

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}, \quad \sinh x \equiv \frac{e^x - e^{-x}}{2}, \quad dc = ds, \quad \cosh^2 x - \sinh^2 x = 1$$

So, for $z = \cosh x$ as above,

$$\text{Arc length} = \int \sqrt{1 + \left(\tfrac{dz}{dx}\right)^2} = \sinh x = n$$

We have shown that $\frac{dz}{dx} = n$, and now we know that our derived n = arc length. Therefore, this equation is equivalent to Equation 1, proving that $z = \cosh x$ is in fact a catenary. The identity $\cosh^2 x - \sinh^2 x = 1$ validates Leibniz's claims about the right triangle laid out in his construction, with sides labeled in his figure as $\overline{\mathrm{OA}}$ and $\overline{\mathrm{AR}}$, wherein $\overline{\mathrm{OA}} = a$, $\overline{\mathrm{AR}} = n$ = arc length, and the length of the hypotenuse is z.[6] The triangle also contains the information needed to construct a tangent line to the catenary at the point (x, z), as explained by Leibniz in Acta Eruditorum.

The argument used above to derive $z = \cosh x$ can be reworked with minor changes to

[6] Leibniz, "De linea in quam flexile se pondere curvat," cited above.

yield the formula for a general catenary:

$$z = a \cosh \frac{x}{a}$$

I conclude with these comments:

The catenary presented by Leibniz in his construction stands out clearly as being based on the exponential function e^x, labeled as his “logarithmic curve.” and his catenary is obviously the hyperbolic cosine not yet named as such. Some properties specified by Leibniz could not have been known without the use of calculus, e.g., the segment constructed equal in length to an arc.

Before knowing about the von Bodenhausen letter, I had wondered what methods of analysis Leibniz used to discover his curve and its properties. It was a mystery for me until I hit upon a simple solution proceding from Equation 1 that reveals the role of the hyperbolic functions explicity, convincing me that the logarithm and exponential function were lying close enough to the surface to be found easily by Leibniz.[7] This note helps to make that more evident.

Leibniz’s construction reveals one of the ways he stood out in the late 17^{th} Century in which the Cartesian rules for defining curves in strict geometric terms were giving way to a newer conception that analytic definitions were also valid, more general and better suited to a wider range of problems in physics.[8]

The dichotomy between geometry vs analysis stands out in the Leibniz construction because it is an impossiblity in the strict sense of a Euclidean ruler-and-compass construction. The fundamental reason is that e is not constructable because it is a transcendetal number; this is the same reason that it is impossible to “square” the circle, i.e., construct a square of

[7] See `http://www.mikeraugh.com/Talks/RIPS-2016-LeibnizCatenary.pdf`.

[8] Bos, Henk J. M., *Redefining Geomterical Exactness: Descarts’ Transformation of the Early Modern Concept of Construction*, Springer, 2001.

equal area, since π is transcendental.

Nevertheless, the Leibniz construction is *theoretically* possible because it is based on two *given* line segments (D and K in one of the representations of his figure). Given segments D and K of the right ratio, his construction can proceed by strict use of a straightedge-and-compass, and in fact Leibniz made the necessary *deus ex machina* assumption equivalent to $d/k = e$. This could trouble geometers of the Descartes school, but not an analyst like Leibniz who accepted the tools of calculus as valid instruments for defining curves. Leibniz had found his catenary using analysis then presented it geometrically.

Appendix A

Leibniz's derivation of Equation 2 from Equation 1 is a good example of the use of differentials in his calculus. Before summarizing Leibniz, we will use the modern idiom of calculus to show that Equation 2 is a consequence of Equation 1.

Modern derivation. Taking into account that Equation 1 assumes that the lowpoint of the catenary is tangent to the x-axis, begin with:

$$\frac{dy}{dx} = y' = \frac{n}{a}. \qquad (y(0) = y'(0) = 0) \tag{A.1}$$

Differentiation yields,

$$y'' = \frac{n'}{a} = \frac{1}{a} \cdot \sqrt{1 + y'^2}. \tag{A.2}$$

Squaring, differentiating and cancelling the redundant y'',

$$y''' = \frac{y'}{a^2} \tag{A.3}$$

Then integrating,

$$y'' = \frac{y}{a^2} + C \tag{A.4}$$

and comparing with Equation A.2 with simplifying algebra, arrive at,

$$(\frac{y}{a^2} + C)^2 = \frac{1}{a^2} \cdot (1 + y'^2) \tag{A.5}$$

The initital conditions of Equation A.2 imply that $C = \frac{1}{a}$. Feeding this back into Equation A.5, we wind up with a result equivalent to Equation 2:

$$y'^2 = y^2 + 2ax \tag{A.6}$$

You will see next that the preceding derivation is essentially what Leibniz does in his own idiom.

Leibniz's derivation. Before proceeding with the derivation of Equation 2 from Equation 1 in his own idiom, it will be useful to illustrate some of the basic techniques Leibniz used. Liebniz postulated that differentials could be manipulated algebraically like actual numerical variables, although he understood them as a useful fiction.[1]

Suppose that a curve is represented at points by an abscissa x and an ordinate y, in which the two are coordinated by analytic expressions, say $f(x)$ and $g(y)$, such that $g(y) = f(x)$. If the respective differentials are, say, $d\,g(y) = g'(y)\,dy$ and $d\,f(x) = f'(x)\,dx$ — where g' and f' are supposed known — then the latter equality fixes the *relationship* between dy and dx at each point (x, y) of the curve but does not determine an *absolute* value for either dx or dy. This allowed Leibniz to specify one or the other for convenience, as will be illustrated below.

For example, suppose $y = x^3$. Then $dy = 3x^2\,dx$. This equation itself can be differentiated using the product rule: $ddy = 6x(dx)^2\,dx + 3x^2\,ddx$.[2] We could choose dx to be constant — while still supposing that the equation $dy = 3x^2\,dx$ holds — and get the simpler result that

[1]For a discussion in detail, see H. J. M. Bos, "Differentials, Higher-Order Differentials and the derivative in the Leibnizian Calculus," in particular Sec. 2.19, *Archive for History of Exact Sciences*, Vol. 14, Springer.

[2]The product rule: $d\,(uv) = u\,dv + v\,du$. Since dy is a legitimate variable, it has a differential: $d\,du = dd\,u$. The symbol dd indicates the differential of a differential, and so on for more-often repeated differentials, e.g., $ddd\,u = d\,(ddu\,)$.

$ddy = 6x\,(dx)^2$. The process can be reversed by taking the "anti-differential" of both sides, recognizing that an undeternined constant must be restored after having been eliminated by the differentiation: $dy = 2x^3dx + Cdx$, where a constant factor of the form Cdx (in which C is an ordinary number) was presumed present in the original expression and needed to be restored.[3]

Here's another application that yields the differential of a quotient. Let $w = \frac{u}{v}$. Multiply by v and differentiate:

$$du = d\,wv = w\,dv + v\,dw = \frac{u}{v} \cdot dv + vdw$$

.

The first and last members of the preceding equations imply that.

$$dw = d\,\frac{u}{v} = \frac{vdu - udv}{v^2}$$

.

Now we return to the problem of deriving Equation 2 from Equation 1 using Leibniz's own idiom. Without loss of generality, simplify Equation 1 by letting $a = 1$ so that we may begin with $\frac{dy}{dx} = n$ and, by definition, $n = \int \sqrt{(dx)^2 + (dy)^2}$. Leibniz chose dx to be a constant. We treat these two equations as follows.

1) Differentiate the first equation (regarding dx as constant), to get: $ddy = dn\,dx$.

2) Differentiate the second equation, square it, and differentiate again (holding dx constant), to get: $dn\,ddn = dy\,ddy$.

[3]So why not some higher power of dx, for example $C(dx)^3$? This appears to be heuristic in Leibniz in consideration of experience differentiating expressions of various types in which each separate act of differentiation adds just one d to each term of the resulting expression. For the case in point, only one d appears in $dy = 3x^2\,dx$, and so only one d should appear in the "constant" term Cdx. This idea works out in practice to produce correct results, and so the finer points of logic can be postponed for another generation of mathematicians.

Combine these two results to eliminate dn, arriving at: $ddn = dy\,dx$. Here Leibniz takes the anti-differential of both sides to get: $dn = y\,dx + C dx$, where C is a constant real number to be determined by initial conditions. By squaring both sides and using the definition of n again deduce that,

$$(dx)^2 + (dy)^2 = (y^2 + 2y + C^2)\,(dx)^2 \qquad \Longrightarrow \qquad y' = \frac{dy}{dx} = \sqrt{y^2 + 2y + C^2 - 1}$$

The initial conditions $y(0) = y'(0) = 0$ give us $C = 1$ and a result equivalent to Equation 2 without the scaling factor a:

$$\frac{dy}{dx} = \sqrt{y^2 + 2y}.$$

The reader may re-introduce a at the beginning of the derivation and derive Equation 2 exactly. The fact that the square-root can be taken in either sense — positive or negative — reflects the fact that the catenary is bilaterally symmetric.